

Optimal Distributed Voltage Regulation in Power Distribution Networks

Albert Y.S. Lam, *Member, IEEE*, Baosen Zhang, *Student Member, IEEE*,
Alejandro Domínguez-García, *Member, IEEE*, and David Tse, *Fellow, IEEE*

Abstract

In this paper, we address the problem of voltage regulation in power distribution networks with deep-penetration of distributed energy resources (DERs), e.g., renewable-based generation, and storage-capable loads such as plug-in hybrid electric vehicles. We cast the problem as an optimization program, where the objective is to minimize the losses in the network subject to constraints on bus voltage magnitudes, limits on active and reactive power injections, transmission line thermal limits and losses. We provide sufficient conditions under which the optimization problem can be solved via its convex relaxation. Using data from existing networks, we show that the conditions are expected to be satisfied by most networks. We also provide an efficient distributed algorithm to solve the problem. The algorithm is asynchronous, with a communication topology that is the same as the electrical network topology. We illustrate the algorithm's performance in the IEEE 34-bus and the 123-bus feeder test systems.

I. INTRODUCTION

Electric power distribution systems will undergo radical transformations in structure and functionality due to the advent of initiatives like the US DOE *Smart Grid*, and its European counterpart *Electricity Networks of the Future*. These transformations are enabled by the integration of i) advanced communication and control, ii) renewable-based variable generation resources, e.g., photovoltaics (PVs), and iii) new storage-capable loads, e.g., plug-in hybrid electric vehicles (PHEVs); these distributed generation and storage resources are commonly

A.Y.S. Lam, B. Zhang and D. Tse are with the EECS Department of the University of California at Berkeley. E-mails: {ayslam,zhangbao,dtse}@eecs.berkeley.edu

A. Domínguez-García is with the Department of ECE Department of the University of Illinois at Urbana-Champaign. E-mail: aledan@illinois.edu

referred to as distributed energy resources (DERs). It has been acknowledged (see, e.g., [1]) that massive penetration of DERs in distribution networks is likely to cause voltage regulation problems due to the fact that typical ratios of line reactance to resistance are such that bus voltage magnitudes are fairly sensitive to variations in active power injections (see, e.g., [2]). Similarly massive penetration of PHEVs can potentially create substantial voltage drops [3]. The objective of this paper is to address the problem of voltage regulation in distribution networks with deep penetration of DERs.

As of today, voltage regulation in distribution networks is accomplished through tap-changing under-load transformers, set voltage regulators, and fixed/switched capacitors. However, these equipments are not designed to deal with rapid voltage fluctuations induced by variations in active power generated and/or consumed by DERs. An alternative to the use of these conventional voltage regulation devices is to utilize the power electronics interface of the DERs themselves to provide reactive power for voltage control, e.g., power electronics-interfaced PV solar rooftop installations in residential and commercial buildings. While the primary function of the power electronics-based interface is to control active power injections, when properly controlled, they can also provide reactive power; thus, they provide a mechanism to control reactive power injections, which in turn can be used for voltage control (see, e.g., [4], [5]). In this regard, there are already existing PV rooftop and pole-mount solutions that provide such functionality (see, e.g., [6], [7]); these solutions are endowed with wireless [6], and power-line communication devices [7], which is a key for controlling a large number of devices without having to overlay a separate communication network. Additionally, as discussed earlier, the voltage profile of a distribution network is sensitivity to changes in active power injections; thus, storage-capable DERs, e.g., PHEVs, and demand response resources (DRRs), provide a second voltage control mechanism as they can be used, to some extent, to shape active power injections.

In this paper, we propose a method to solve the voltage regulation problem in distribution networks that relies on the utilization of reactive-power capable DERs, and to some extent, on the control of active power injections enabled by storage-capable DERs and DRRs. We cast the problem as an optimization program where the objective is to minimize the losses in the network subject to i) constraints on bus voltage magnitudes, ii) upper and lower limits on active and reactive power injections, iii) upper limits on transmission line flows, and iv) upper limits on transmission line losses. The decision variables are the bus voltages; however the actual control

mechanism to fix these are active and reactive power injections on each bus. This optimization problem can be written as a rank-constrained semidefinite program (SDP [8]) as observed in [9], where the decision variable is a positive semidefinite matrix constrained to have rank 1, rendering, in general a non-convex problem. The problem can be convexified by dropping the rank-1 constraint; the conundrum is to establish when the solution of the convexified problem provides also a global solution to the original non-convex problem.

Recently there has been a sequence of papers on attempting to answer this question spurred by the observation in [10] that the convex relaxation observed above is tight for many IEEE benchmark networks. Several independent works [11]–[13] provided a partial answer: the convex relaxation is tight if the network has a *tree* topology and certain constraints on the bus power injections are satisfied. These results were unified and strengthened in [14] through a investigation of the underlying geometry of the optimization problem.

The first contribution of this paper is in extending the results in [14] to include reactive powers, providing i) sufficient conditions under which the optimal solution of the convexified problem is also the solution of the original problem, and ii) a simple method to establish whether or not the original problem has a feasible solution based on the solution of the convexified problem. However, it is important to note that even if the convexified SDP provides the solution to the original optimization problem, existing algorithms for solving SDPs are not computationally efficient for solving large problems [8]; therefore these algorithms are not practical for realizing our ideas in a realistic power distribution network with thousands of buses. Furthermore even if there is a centralized solver with sufficient computing power, the communication infrastructure may not be able to reliably transmit all problem data to one particular location with small enough delays; this is where the second contribution of this paper lies—the development of an asynchronous distributed algorithm for efficiently solving convexified SDPs on tree networks.

There is previous work that has addressed the voltage control problem by casting it as an optimization problem. For example [15] proposes the use of reactive-power capable DERs for voltage control; the objective is to minimize the DER reactive power contributions subject to the power flow equations and other constraints; and the solution approach relies on linearizing the power flow equations around some operating point, rendering a linear program. In [16], the authors also use reactive-power capable DERs for voltage control. They partition the system buses into groups and provide an algorithm to solve the optimization problem by breaking it

into subproblems, each of which is associated with a particular group of buses—the solution that this algorithm provides is not necessarily the globally optimal one. In [17], a centralized decision maker maintains a database of historical observations and the corresponding solutions to the optimization problem obtained offline.

The remainder of this paper is organized as follows. In Section II, we formulate the voltage regulation problem as an optimization problem. Section III states the main theoretical result of the paper, while Section IV provides a sketch of the proof (the complete proof is provided in the Appendix). Section IV may be skipped without loss of continuity for the reader mainly interested in the algorithm. Section V proposes a distributed algorithm to solve the voltage control problem, the performance of which is illustrated in Section VI. Concluding remarks are presented in Section VII.

II. PRELIMINARIES AND PROBLEM FORMULATION

In this section, we first introduce the power network model, including relevant notation used throughout the paper, and then formulate the voltage regulation problem.

A. Power System Distribution Model

Consider a power distribution network with $1, 2, \dots, n$ buses. As of today, such networks are mostly radial with a single source of power injection referred to as the feeder (see, e.g., [18]). Thus, the network topology can be described by a *connected tree*, the edge set of which is denoted by \mathcal{E} , where $(i, k) \in \mathcal{E}$ if i is connected to k by a transmission line; we write $i \sim k$ if bus i is connected to bus k and $i \not\sim k$ otherwise, and (i, k) to denote a line connected between i and k .

Let $V_i = |V_i| \angle \theta_i$ describe the voltage in bus i and define the corresponding bus voltage vector as $\mathbf{v} = [V_1 \ V_2 \ \dots \ V_n]^T \in \mathbb{C}^n$. Similarly, let P_i and Q_i denote, respectively the active and reactive power injections in bus i , and define the corresponding active and reactive power injection vectors as $\mathbf{p} = [P_1 \ P_2 \ \dots \ P_n]^T$, $\mathbf{q} = [Q_1 \ Q_2 \ \dots \ Q_n]^T$. Let $y_{ik} = g_{ik} - jb_{ik}$, with $b_{ik} > g_{ik} > 0$, denote the admittance¹ of line (i, k) , and let $y_{ii} = jb_{ii}$ denote bus i shunt admittance. Then, the power

¹We adopt the standard assumption that, in normal operating conditions, lines are inductive and the inductive effects dominate resistive effects (see, e.g., [19]). Additionally, although the convention is to write $y_{ik} = g_{ik} + jb_{ik}$, we flip the sign of the imaginary part as it simplifies the developments.

flow equations can be compactly written as

$$\mathbf{p} + j\mathbf{q} = \text{Re}\{\text{diag}(\mathbf{v}\mathbf{v}^H\mathbf{Y}^H)\} + j\text{Im}\{\text{diag}(\mathbf{v}\mathbf{v}^H\mathbf{Y}^H)\}, \quad (1)$$

where $\mathbf{Y} = [Y_{ik}]$ is the bus admittance matrix (sometimes we use $\mathbf{A}[i, k]$ to denote the $(i, k)^{th}$ entry of a matrix \mathbf{A}), \mathbf{v}^H (\mathbf{Y}^H) denotes the Hermitian transpose of \mathbf{v} (\mathbf{Y}), and $\text{diag}(\cdot)$ returns the diagonal of a square matrix as a column vector. The active/reactive power flows through each line (i, k) are

$$P_{ik} = |V_i|^2 g_{ik} + |V_i||V_k|[b_{ik} \sin(\theta_{ik}) - g_{ik} \cos(\theta_{ik})], \quad (2)$$

$$Q_{ik} = |V_i|^2 b_{ik} - |V_i||V_k|[g_{ik} \sin(\theta_{ik}) + b_{ik} \cos(\theta_{ik})], \quad (3)$$

where $\theta_{ik} := \theta_i - \theta_k$.

B. Voltage Regulation Problem Formulation

We are interested in keeping each individual bus voltage magnitude $|V_i|$ at some prescribed value V_i^{ref} imposed by performance requirements. In order to achieve so, we rely on a limited ability to control active/reactive power injections—this is a realistic scenario in a distribution system with deep penetration of renewable-based generation, PHEVs and DRRs—and cast the problem as an optimization program where the objective is to minimize the losses in the network.

Let $L_{ik}(V_i, V_k) = P_{ik} + P_{ki}$; then, the total losses in the network are given by $L(\mathbf{v}) := \sum_{i,k:(i,k) \in \mathcal{E}} L_{ik}(V_i, V_k)$ and the voltage regulation problem can be formulated as

$$\underset{\mathbf{v}}{\text{minimize}} \quad L(\mathbf{v}) \quad (4a)$$

$$\text{subject to } |V_i| = V_i^{ref}, \forall i \quad (4b)$$

$$\underline{P}_i \leq P_i \leq \overline{P}_i, \forall i \quad (4c)$$

$$\underline{Q}_i \leq Q_i \leq \overline{Q}_i, \forall i \quad (4d)$$

$$|P_{ik}| \leq \overline{P}_{ik}, \forall i \sim k \quad (4e)$$

$$L_{ik}(V_i, V_k) \leq \overline{L}_{ik}, \forall i \sim k, \quad (4f)$$

where P_i and Q_i are functions of \mathbf{v} as defined in (1), and P_{ik} is a function of V_i and V_k as defined in (2). Without loss of generality and to ease the notations in subsequent developments, hereafter we assume $V_i^{ref} = 1$ p.u. for all i . The constraints in (4b) capture the voltage regulation

goal. The constraints in (4c) and (4d) describe the limited ability to control active/reactive power injections on each bus i ; \overline{P}_i (\underline{P}_i), and \overline{Q}_i (\underline{Q}_i), respectively denote the upper (lower) limits on the amount of active and reactive power that each bus i can provide. The constraints in (4e) capture line thermal limits, while (4f) imposes loss limits on individual lines, this ensures an homogeneous distribution of losses across the system.

The optimization problem in (4) is difficult for two reasons: i) it is not convex due to the quadratic relationship between bus voltages and powers; ii) depending on the size of the network, there could potentially be a large number of variables and constraints. Section III and Section IV address i) by convexifying the problem, while Section V addresses ii) by proposing a computationally efficient algorithm to solve it.

III. CONVEX RELAXATION

In this section, we state the main theoretical result, which is that under certain conditions on the angle differences between adjacent buses and the lower bounds on reactive power injections, the nonconvex problem in (4) can be solved exactly by solving its convex SDP relaxation. In order to state the exact result, it is convenient to write (4) in matrix form.

A. Voltage Regulation Problem Formulation in Matrix Form

Let $\mathbf{E}_i \in \mathbb{R}^{n \times n}$, with $E_{ii} = 1$ and all other entries equal to zero, and define $\mathbf{A}_i = \frac{1}{2}(\mathbf{Y}^H \mathbf{E}_i + \mathbf{E}_i \mathbf{Y})$, and $\mathbf{B}_i = \frac{1}{2j}(\mathbf{Y}^H \mathbf{E}_i - \mathbf{E}_i \mathbf{Y})$. Then, the active and reactive power injections in bus i are, respectively, given by $P_i = \text{Tr}(\mathbf{A}_i \mathbf{v} \mathbf{v}^H)$, and $Q_i = \text{Tr}(\mathbf{B}_i \mathbf{v} \mathbf{v}^H)$, where $\text{Tr}(\cdot)$ is the trace operator. For each $(i, k) \in \mathcal{E}$, define a matrix \mathbf{A}_{ik} , with its $(l, m)^{th}$ entry given by

$$\mathbf{A}_{ik}[l, m] = \begin{cases} -\text{Re}\{Y_{ik}\} & \text{if } l = m = i \\ Y_{ik}/2 & \text{if } l = i \text{ and } m = k \\ Y_{ik}^H/2 & \text{if } l = k \text{ and } m = i \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

the active power flow through the (i, k) line is given by $P_{ik} = \text{Tr}(\mathbf{A}_{ik} \mathbf{v} \mathbf{v}^H)$. Let $\mathbf{G}_{ik} = \mathbf{A}_{ik} + \mathbf{A}_{ki}$. Then, we can rewrite (4) as

$$\underset{\mathbf{v}}{\text{minimize}} \quad \sum_{i=1}^n \text{Tr}(\mathbf{A}_i \mathbf{v} \mathbf{v}^H) \quad (6a)$$

$$\text{subject to } |V_i| = 1, \forall i \quad (6b)$$

$$\underline{P}_i \leq \text{Tr}(\mathbf{A}_i \mathbf{v} \mathbf{v}^H) \leq \overline{P}_i, \forall i \quad (6c)$$

$$\underline{Q}_i \leq \text{Tr}(\mathbf{B}_i \mathbf{v} \mathbf{v}^H) \leq \overline{Q}_i, \forall i \quad (6d)$$

$$\text{Tr}(\mathbf{G}_{ik} \mathbf{v} \mathbf{v}^H) \leq \overline{L}_{ik}, \forall i \sim k \quad (6e)$$

$$|\text{Tr}(\mathbf{A}_{ik} \mathbf{v} \mathbf{v}^H)| \leq \overline{P}_{ik}, \forall i \sim k. \quad (6f)$$

Note the outer product $\mathbf{v} \mathbf{v}^H$ is a positive semidefinite rank-1 $n \times n$ matrix. Conversely, given a positive semidefinite rank-1 $n \times n$ matrix, it is always possible to write it as an outer product of a vector and itself. Thus, we can rewrite (6) as

$$\underset{\mathbf{W} \succeq 0}{\text{minimize}} \quad \sum_{i=1}^n \text{Tr}(\mathbf{A}_i \mathbf{W}) \quad (7a)$$

$$\text{subject to } |\mathbf{W}[i, i]| = 1, \forall i \quad (7b)$$

$$\underline{P}_i \leq \text{Tr}(\mathbf{A}_i \mathbf{W}) \leq \overline{P}_i, \forall i \quad (7c)$$

$$\underline{Q}_i \leq \text{Tr}(\mathbf{B}_i \mathbf{W}) \leq \overline{Q}_i, \forall i \quad (7d)$$

$$\text{Tr}(\mathbf{G}_{ik} \mathbf{W}) \leq \overline{L}_{ik}, \forall i \sim k \quad (7e)$$

$$|\text{Tr}(\mathbf{A}_{ik} \mathbf{W})| \leq \overline{P}_{ik}, \forall i \sim k \quad (7f)$$

$$\text{rank}(\mathbf{W}) = 1; \quad (7g)$$

the rank-1 constraint in (7g) renders the problem non-convex.

B. Convexification

The problem in (7) is not convex due to the in constraint (7g); by removing it, we obtain a relaxation that is convex:

$$\underset{\mathbf{W} \succeq 0}{\text{minimize}} \quad \sum_{i=1}^n \text{Tr}(\mathbf{A}_i \mathbf{W}) \quad (8a)$$

$$\text{subject to } |\mathbf{W}[i, i]| = 1, \forall i \quad (8b)$$

$$\underline{P}_i \leq \text{Tr}(\mathbf{A}_i \mathbf{W}) \leq \overline{P}_i, \forall i \quad (8c)$$

$$\underline{Q}_i \leq \text{Tr}(\mathbf{B}_i \mathbf{W}) \leq \overline{Q}_i, \forall i \quad (8d)$$

$$\text{Tr}(\mathbf{G}_{ik} \mathbf{W}^H) \leq \overline{L}_{ik}, \forall i \sim k \quad (8e)$$

$$|\text{Tr}(\mathbf{A}_{ik} \mathbf{W})| \leq \overline{P}_{ik}, \forall i \sim k \quad (8f)$$

The convex relaxation is not always tight since the rank of the solution to (8) could be greater than 1. Thus the solution to (8) does not always coincide with the solution to (7). However, by imposing two conditions that are widely held in practice, the non-convex problem in (7) can be solved exactly by solving (8) as stated in the following theorem.

Theorem 1. Define θ_{ik}^P to be the smallest positive solution to the equation $\overline{P}_{ik} = P_{ik}$, with P_{ik} given in (2) for $|V_i| = |V_k| = 1$; and $\theta_{ik}^L = \cos^{-1}(1 - \frac{\overline{L}_{ik}}{2g_{ik}})$ if $\frac{\overline{L}_{ik}}{2g_{ik}} \leq 2$ and $\theta_{ik}^L = \infty$ if $\frac{\overline{L}_{ik}}{2g_{ik}} > 2$. Let $\bar{\theta}_{ik} = \min(\theta_{ik}^P, \theta_{ik}^L)$. Suppose $\bar{\theta}_{ik}$ satisfies

$$-\tan^{-1}(b_{ik}/g_{ik}) < \bar{\theta}_{ik} < \tan^{-1}(b_{ik}/g_{ik}); \quad (9)$$

and reactive power injection lower bounds satisfy

$$\overline{Q}_i < \beta_i, \quad i = 2, \dots, n, \quad (10)$$

with $\beta_i = \sum_{k:k \in \mathcal{C}(i)} b_{ik} - g_{ik} \sin(\tilde{\theta}_{ik}) - b_{ik} \cos(\tilde{\theta}_{ik})$, where $\mathcal{C}(i)$ is the set of all neighbors of i and $\tilde{\theta}_{ik} = \min(\tan^{-1}(\frac{g_{ik}}{b_{ik}}), \bar{\theta}_{ik})$. Let \mathbf{W}^* be an optimal solution to the relaxed problem (8). Then

- 1) If \mathbf{W}^* is rank 1, then $\mathbf{W}^* = \mathbf{v}^*(\mathbf{v}^*)^H$ for some vector \mathbf{v}^* . Furthermore, \mathbf{v}^* is the optimal solution to the voltage regulation problem stated in (4).
- 2) If $\text{rank}(\mathbf{W}^*) > 1$, then there is no feasible solution to the voltage regulation problem stated in (4).
- 3) If (8) is infeasible, then the voltage regulation problem stated in (4) is infeasible.

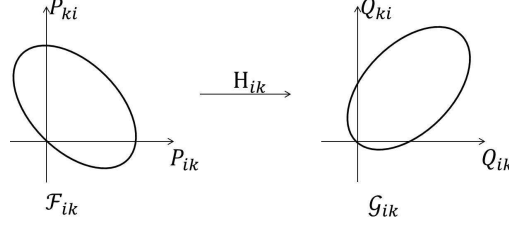


Fig. 1. The active line flow region \mathcal{F}_{ik} , the reactive flow region \mathcal{G}_{ik} , and the linear transformation \mathbf{H}_{ik} between them.

The theorem is proved for a two-bus network in Section IV by studying the geometry of the feasibility set of the original problem in (6) and that of its convex relaxation in (8). The intuition and geometric insight developed by studying the two-bus network carries over to a general tree network and the full proof is provided in the Appendix.

IV. SKETCH OF THEOREM 1 PROOF

The insights into Theorem 1 are obtained by studying the geometry of the sets that result from the constraints on line power flows and power injections as described in (4c)–(4f). This geometric view was explored in previous works [11], [14]. Here, we revisit the results of [14] and generalize them to include reactive power.

A. Active and reactive line flow regions

First, recall from (7b) that $|V_i| = |V_k| = 1$ p.u. Then, let $\mathcal{F}_{ik} \in \mathbb{R}^2$ and $\mathcal{G}_{ik} \in \mathbb{R}^2$, respectively, denote the regions that contain all the $[P_{ik}, P_{ki}]^T$ and $[Q_{ik}, Q_{ki}]^T$ that can be achieved from (2) and (3) by varying θ_{ik} between 0 and 2π ; it is easy to see that for $0 < \theta_{ik} < 2\pi$, (2) and (3) are linear transformations of a circle. Thus, as depicted in Fig. 1, the active and reactive line flow regions \mathcal{F}_{ik} and \mathcal{G}_{ik} are two ellipses. The center of \mathcal{F}_{ik} (\mathcal{G}_{ik}) is $[g_{ik}, g_{ik}]^T$ ($[b_{ik}, b_{ik}]^T$), its major axis is parallel to $[1, -1]^T$ ($[1, 1]^T$) and has length b_{ik} (b_{ik}), while its minor axis is parallel to $[1, 1]^T$ ($[1, -1]^T$) and has length g_{ik} (g_{ik}). Both ellipses are related by a linear invertible mapping: $\mathcal{G}_{ik} = \mathbf{H}_{ik}\mathcal{F}_{ik}$, with

$$\mathbf{H}_{ik} = \frac{1}{2b_{ik}g_{ik}} \begin{bmatrix} b_{ik}^2 - g_{ik}^2 & b_{ik}^2 + g_{ik}^2 \\ b_{ik}^2 + g_{ik}^2 & b_{ik}^2 - g_{ik}^2 \end{bmatrix}. \quad (11)$$

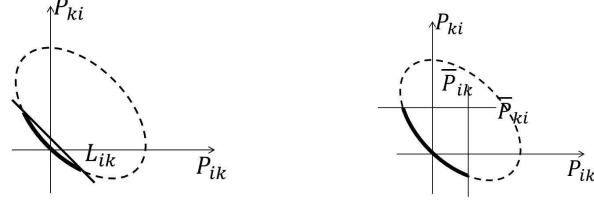


Fig. 2. The flow region under thermal loss constraints (left) and line flow constraints (right). The bold curves indicate the feasible part.

The line flow constraints in (4e) and the thermal loss constraints in (4f) appear as linear constraints on the line flow regions as shown in Fig. 2. Thus, for each line (i, k) , we can replace both constraints by a single one, which has the form of the line flow constraint for properly defined upper limits. We adopt this convention in subsequent developments. Furthermore, since the ellipses have empty interior, this flow constraint can be translated into angle constraints on θ_{ik} of the form $|\theta_{ik}| \leq \bar{\theta}_{ik}$. Conversely, an angle constraint on θ_{ik} can be converted into a flow constraint. Let $\mathcal{F}_{\theta,ik}$ and $\mathcal{G}_{\theta,ik}$ denote, respectively, line (i, k) angle-constrained active and reactive line flow regions, then

$$\begin{aligned}\mathcal{F}_{\theta,ik} &= \{[P_{ik}, P_{ki}]^T : P_{ik} = g_{ik}[1 - \cos(\theta_{ik})] + b_{ik} \sin(\theta_{ik}), \\ &\quad P_{ki} = g_{ik}[1 - \cos(\theta_{ik})] - b_{ik} \sin(\theta_{ik}), |\theta_{ik}| \leq \bar{\theta}_{ik}\}, \\ \mathcal{G}_{\theta,ik} &= \{[P_{ik}, P_{ki}]^T : P_{ik} = b_{ik}[1 - \cos(\theta_{ik})] - g_{ik} \sin(\theta_{ik}), \\ &\quad P_{ki} = b_{ik}[1 - \cos(\theta_{ik})] + g_{ik} \sin(\theta_{ik}), |\theta_{ik}| \leq \bar{\theta}_{ik}\}.\end{aligned}$$

B. Feasible Region of a Two-Bus Network

Consider a system with only two buses connected by a line $(1, 2)$, with P_1 (Q_1), and P_2 (Q_2) denoting the active (reactive) power injections on bus 1 and 2, respectively; for a two-bus system, $P_1 = P_{12}$ ($Q_1 = Q_{12}$) and $P_2 = P_{21}$ ($Q_2 = Q_{21}$). The relaxed problem in (8) convexifies the feasible region of the problem (7) by filling up the corresponding ellipses as shown in Fig. 3. The condition in (9) is a constraint on the maximum angle difference across the line. Intuitively, this angle constraint is such that only the lower left part of the line flow ellipses is feasible.

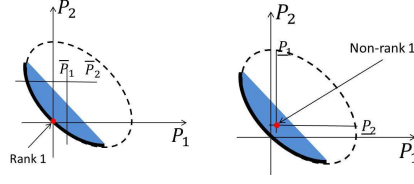


Fig. 3. Angle-constrained active line flow region and its convex hull (filled in blue region) when relaxation is tight (left) and when relaxation does not provide solution to the original problem (right).

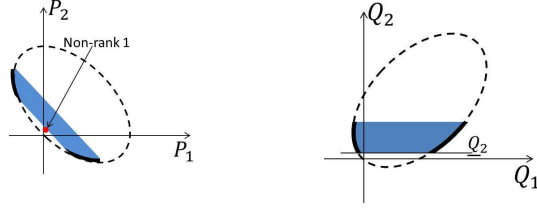


Fig. 4. Active power injection region (left) and reactive power injection region (right) under reactive power injection lower bound.

For example, this condition is satisfied by the angle-constrained regions in Fig. 3. Fig. 3 shows the intersection of bus power constraints with the angle-constrained active injection regions. In Fig. 3(a), both bus power constraints are upper bounds. Since the optimal solution of the power loss problem occurs in the lower left corner, the convex relaxation is tight. In Fig. 3(b), both bus power constraints are lower bounds; in this case the optimal solution is inside the ellipses, therefore $\text{rank } \mathbf{W}^* = 2$ and the original problem is infeasible.

It is important to note that the observations made in Fig. 3 hold as long as the angle-constrained injection region only includes the lower left half of the ellipse (as described by (9)). From thermal data for some common lines in [18], we expect that the angle to be constrained to $\theta_{ik} \in [-10^\circ, 10^\circ]$. Even for a relatively small b_{ik}/g_{ik} ratio of 2, $\bar{\theta}_{ik} = \tan^{-1}(b_{ik}/g_{ik}) = 63.4^\circ$ and the condition $|\theta_{ik}| < \bar{\theta}_{ik}$ is always satisfied. Therefore in most practical networks, it is expected that the thermal constraints in the network is small enough that the condition in (9) should be satisfied almost always.

The second condition in (10) is to ensure that the reactive lower bound is large enough such that $Q_i > \underline{Q}_i$ for all feasible Q_i . If the reactive lower bounds are tight at the optimal solution of

the relaxed problem, then the rank of the optimal matrix \mathbf{W}^* is not necessarily 1. Figure 4 shows the reason that the condition on the reactive power lower bounds are needed. Figure 4(b) gives the reactive injection region with a tight reactive lower bound on bus 2. Figure 4(a) shows the corresponding active power injection region. Observe that it is possible for the optimal solution of the relaxed problem to be rank 2, while the original problem remains feasible. The condition $\overline{Q}_i < \beta_i$ rules out this phenomenon by ensuring that the reactive power lower bounds are never tight.

C. General Tree Networks

The geometrical intuition developed for the two-bus network carries over to a general tree network due to two reasons: the flows on each line are independent (no cycles) and the active and reactive power injections can be described, respectively, as linear combinations of the active and reactive line flows; these are the main ideas used in the proof of Theorem 1 (see the Appendix).

V. A DISTRIBUTED ALGORITHM FOR SOLVING THE CONVEXIFIED PROBLEM

In Section III we showed that the SDP program in (8) is a convex relaxation of the voltage support problem in (4). Since the objective is to regulate the voltages in the presence of renewable sources, the optimization problem needs to be solved no slower than the time scale at which the renewable-based power injections change significantly. General-purpose SDP solvers scale poorly as the problem size increases [8]. Thus for large distribution networks with hundreds or thousands of buses, solving the SDP problem in a minute to sub-minute scale is challenging. Furthermore standard solvers for SDP problems are centralized; i.e., all problem data are assumed to be presented at a single processor. However the communication infrastructure in a distribution network may not be able to transmit the data from all users to a centralized location fast enough. By exploiting the tree structure of the distribution network, we propose a distributed and asynchronous algorithm to solve (8) that only requires communication between two neighboring buses in the network. An independent but related work has appeared in [20].

A. Algorithm Derivation

The proposed algorithm consists of two stages: *local optimization* and *consensus*. In the local optimization stage, each node solves its own local version of the problem. In the consensus

stage, neighboring nodes exchange Lagrangian multipliers obtained from the solutions to their corresponding local optimums, with the goal of equalizing the phase angle differences across a line from both of its ends.

Let \mathcal{N}_i be the set of buses directly connected to bus i by transmission lines, together with bus i itself, i.e., $\mathcal{N}_i = \{k : k \sim i, \forall k\} \cup \{i\}$. For a $n \times n$ matrix \mathbf{M} , let $\mathbf{M}^{(i)}$ denote the $|\mathcal{N}_i| \times |\mathcal{N}_i|$ submatrix of \mathbf{M} whose rows and columns are indexed according to \mathcal{N}_i . Similarly, for the $n \times 1$ vector \mathbf{v} , $\mathbf{v}^{(i)}$ is the corresponding \mathcal{N}_i -dimensional vector indexed by \mathcal{N}_i . We can rewrite (8) as

$$\underset{\mathbf{W}^{(1)}, \dots, \mathbf{W}^{(n)} \succcurlyeq 0}{\text{minimize}} \quad \sum_{i=1}^n \text{Tr}(\mathbf{A}^{(i)} \mathbf{W}^{(i)}) \quad (12a)$$

$$\text{subject to } \text{diag}(\mathbf{W}^{(i)}) = \mathbf{v}^{(i)} \circ \mathbf{v}^{(i)}, \forall i \quad (12b)$$

$$\underline{P}_i \leq \text{Tr}(\mathbf{A}^{(i)} \mathbf{W}^{(i)}) \leq \overline{P}_i, \forall i \quad (12c)$$

$$\underline{Q}_i \leq \text{Tr}(\mathbf{B}^{(i)} \mathbf{W}^{(i)}) \leq \overline{Q}_i, \forall i \quad (12d)$$

$$|\text{Tr}(\mathbf{A}_{ik}^{(i)} \mathbf{W}^{(i)})| \leq \overline{P}_{ik}, \forall (i, k) \in \mathcal{E} \quad (12e)$$

$$W_{ik}^{(i)} = W_{ik}^{(k)}, \forall (i, k) \in \mathcal{E}, \quad (12f)$$

$$W_{ki}^{(i)} = W_{ki}^{(k)}, \forall (i, k) \in \mathcal{E}, \quad (12g)$$

where \circ is the Hadamard product. It is easy to verify that (12a), (12b), (12c), (12d), and (12e) are equivalent to (8a), (8b), (8c), (8d), and (8f), respectively, as \mathbf{A}_i in (8c), \mathbf{B}_i in (8d), and \mathbf{A}_{ik} in (8f) have non-zero elements only at (i, i) , (i, k) , (k, i) , $\forall k \sim i$. Since all \mathcal{N}_i 's are maximal cliques, $\mathbf{W} \succcurlyeq 0$ is tantamount to $\mathbf{W}^{(i)} \succcurlyeq 0, \forall i$ [21], [22]. Constraints (12f) and (12g) are added to ensure that all $\mathbf{W}^{(i)}$'s coordinate to form \mathbf{W} ; in other words, $\forall (i, k) \in \mathcal{E}$, the θ_{ik} 's computed from $\mathbf{W}^{(i)}$ and $\mathbf{W}^{(k)}$ should be the same.

Let λ_{ik} be the Lagrangian multiplier of (12f) for (i, k) and similarly λ_{ki} for (12g). By relaxing (12f) and (12g), the augmented objective function is

$$\begin{aligned} & \sum_{i=1}^n \text{Tr}(\mathbf{A}^{(i)} \mathbf{W}^{(i)}) + \sum_{(i,k) \in \mathcal{E}} [\lambda_{ik}(W_{ik}^{(i)} - W_{ik}^{(k)}) \\ & + \lambda_{ki}(W_{ki}^{(i)} - W_{ki}^{(k)})] \triangleq \sum_{i=1}^n \text{Tr}(\tilde{\mathbf{A}}^{(i)} \mathbf{W}^{(i)}), \end{aligned} \quad (13)$$

where $\tilde{\mathbf{A}}^{(i)}$ is also Hermitian, and its $(i, k)^{th}$ entry is i) $\tilde{A}_{ik}^{(i)} = A_{ik}^{(i)}$ if $i = k$, $\tilde{A}_{ik}^{(i)} = A_{ik}^{(i)} + \lambda_{ik}^H$ if $i < k$, and iii) $\tilde{A}_{ik}^{(i)} = A_{ik}^{(i)} - \lambda_{ik}^H$ if $i > k$. With (13), problem (12) can be divided into n

separable subproblems and the i th subproblem corresponds to bus i , defined as follows:

$$\underset{\mathbf{W}^{(i)} \succcurlyeq 0}{\text{minimize}} \quad \text{Tr}(\tilde{\mathbf{A}}^{(i)} \mathbf{W}^{(i)}) \quad (14a)$$

$$\text{subject to} \quad \text{diag}(\mathbf{W}^{(i)}) = \mathbf{v}^{(i)} \circ \mathbf{v}^{(i)} \quad (14b)$$

$$\underline{P}_i \leq \text{Tr}(\mathbf{A}^{(i)} \mathbf{W}^{(i)}) \leq \overline{P}_i \quad (14c)$$

$$\underline{Q}_i \leq \text{Tr}(\mathbf{B}^{(i)} \mathbf{W}^{(i)}) \leq \overline{Q}_i \quad (14d)$$

$$|\text{Tr}(\mathbf{A}_{ik}^{(i)} \mathbf{W}^{(i)})| \leq \overline{P}_{ik}, \quad \forall k \sim i. \quad (14e)$$

We denote the feasible region described by (14b)–(14e) together with $\mathbf{W}^{(i)} \succcurlyeq 0$ of Subproblem i by \mathcal{C}_i . Define $g_i(\lambda_{ik}) \triangleq \inf_{\mathbf{W}^{(i)} \in \mathcal{C}_i} \{\text{Tr}(\tilde{\mathbf{A}}^{(i)} \mathbf{W}^{(i)})\}$. The gradient of $-g_i$ at λ_{ik} is $W_{ik}^{(i)*}$, which is the (i, k) th element of the optimal $\mathbf{W}^{(i)*}$ of g_i determined by solving the i th subproblem (14). Similarly, that of $-g_k$ at λ_{ik} is $-W_{ik}^{(k)*}$. Therefore, the gradient of $-(g_i + g_k)$ is then $W_{ik}^{(i)*} - W_{ik}^{(k)*}$. Let $W_{ik}^{(i)}[t]$ and $W_{ik}^{(k)}[t]$ be $W_{ik}^{(i)*}$ and $W_{ik}^{(k)*}$ determined at time t , respectively. By gradient ascent, at time $t + 1$, we update λ_{ik} by

$$\lambda_{ik}[t + 1] = \lambda_{ik}[t] + \alpha[t](W_{ik}^{(i)}[t] - W_{ik}^{(k)}[t]), \quad (15)$$

where $\alpha[t] > 0$ and $\lambda_{ik}[t]$ are the step size and λ_{ik} at time t , respectively. The value of $\lambda_{ki}[t + 1]$ can be directly computed from $\lambda_{ik}[t + 1]$ as $\lambda_{ki} = \lambda_{ki}^H$. The Lagrangian multiplier λ_{ik} is only defined for the line (i, k) and the two buses at the ends of the edge, i.e., buses i and k , are required to manipulate λ_{ik} . The purpose of (15) is to make $W_{ik}^{(i)}$ and $W_{ik}^{(k)}$ as close to each other as possible with the help of λ_{ik} . Eq. (15) can be computed either by bus i or by bus k and it is independent of all other buses and edges. Whenever both the i th and k th subproblems have been computed and so $W_{ik}^{(i)}$ and $W_{ik}^{(k)}$ have been updated, then λ_{ik} can then be updated by using (15).

The optimization problem comprised of (13), together with all the constraints (14b)–(14e), imposed on the subproblems, is a dual problem of (12). When all λ_{ik} 's are optimal, $W_{ik}^{(i)}$ will be equal to $W_{ik}^{(k)}$ for all (i, k) 's and thus the duality gap is zero. Accordingly, we can construct the optimal \mathbf{W}^* of problem (7g) from the values of the $W_{ik}^{(k)}$'s. Furthermore, for a $\delta > 0$ sufficiently small, and due to the convexity of (12) and the zero duality gap, Algorithm 1 is guaranteed to converge to the optimum.

Algorithm 1 Asynchronous Distributed Algorithm

Given a n -bus network

1. **while** $|W_{ik}^{(i)} - W_{ik}^{(k)}| > \delta$ for any $(i, k) \in \mathcal{E}$ **do**
 2. **for** each bus i (in parallel) **do**
 3. Given $\lambda_{ik}, \forall k \sim i$, solve (14)
 4. Return $W_{ik}^{(i)}, \forall k$
 5. **end for**
 6. Given $W_{ik}^{(i)}$ and $W_{ik}^{(k)}$, update λ_{ik} with (15) (in parallel and asynchronously)
 7. **end while**
-

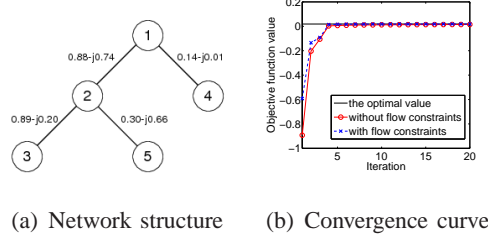


Fig. 5. A five-bus example

B. Feasibility

When the buses determine their own limits on active and reactive powers independently, an infeasible problem might result, i.e., an empty feasible region. When there exists a central authority having all the bus power information, we can check the feasibility easily. Otherwise, it is necessary for the buses to declare infeasibility.

One sufficient condition for infeasibility of the the problem is that there exists an infeasible subproblem (14) for any bus. If any bus finds an infeasible subproblem, it is sufficient to say that the whole problem is infeasible. To proceed further, the bus with an infeasible subproblem should adjust its own active and reactive power limits so as to make the subproblem feasible. A necessary and sufficient condition for infeasibility is that $W_{ik}^{(i)}$ and $W_{ik}^{(k)}$ never match for some $(i, k) \in \mathcal{E}$ when Algorithm 1 evolves. If this happens on edge (i, k) , either bus i or bus k or both constitute the infeasibility.

TABLE I
BUS INFORMATION OF THE FIVE-BUS EXAMPLE

Bus	\bar{P}	\underline{P}	\bar{Q}	\underline{Q}	\bar{V}
1	5.2844	-5.4692	5.5798	-5.7604	1.2247
2	-0.0648	-0.0988	0.5298	0	1.1509
3	-0.0423	-0.5828	0.6405	0	1.1103
4	-0.0334	-0.5155	0.2091	0	0.9762
5	-0.0226	-0.4329	0.3798	0	1.1400

C. Numerical Performance Enhancements

Consider the five-bus network given in Fig. 5(a). Assuming that all λ_{ik} 's are updated at the end of each iteration, the progress of Algorithm 1 (the curve without power flow constraints) and the target optimal objective value are shown in Fig. 5(b). At iteration 20, when we sum the objective function values of all the subproblems, the sum still has around 20% difference to the optimal one. Even for a small network, it may take a long time for the algorithm to converge to the global optimal solution. Next, we provide some enhancements that improve the algorithm convergence speed.

1) Power Flow Constraints: Constraint (14e) means that the active power can flow in any direction on the edge (i, k) as long as its magnitude does not exceed the limit \bar{P}_{ik} . Assume that the global optimal solution \mathbf{W}^* exists. Our decomposition allows us to compute W_{ik}^* separately by buses i and k , in which each bus determines its local version of W_{ik}^* , e.g., $W_{ik}^{(i)}$ for bus i . Then (15) brings both $W_{ik}^{(i)}$ and $W_{ik}^{(k)}$ towards W_{ik}^* by just equalizing $W_{ik}^{(i)}$ and $W_{ik}^{(k)}$. If the feasible regions \mathcal{C}_i and \mathcal{C}_k are smaller, it will be easier for (15) to reduce the discrepancy between $W_{ik}^{(i)}$ and $W_{ik}^{(k)}$.

The additional assumption we make is that all buses are net consumers of active power except the feeder; that is, $P_i \leq 0$ for $i = 2, 3, \dots, n$. It is because even for deep penetration levels of renewable-based electricity generation in distribution systems, it is unlikely that the power generated by these resources in a particular bus will compensate the load at that bus; let alone they will compensate for the power coming from the feeder. We now know that the active power must flow from buses i to k along the edge (i, k) with $i < k$, i.e., $P_{ik} \geq 0$. With this observation,

we can re-write (14e) as

$$0 \leq P_{ik} = \text{Tr}(\mathbf{A}_{ik}^{(i)} \mathbf{W}^{(i)}) \leq \bar{P}_{ik}, \quad (16)$$

$$-\bar{P}_{ik} \leq P_{ki} = \text{Tr}(\mathbf{A}_{ik}^{(k)} \mathbf{W}^{(k)}) \leq 0, \quad (17)$$

from the perspectives of buses i and k , respectively. We can actually replace (14e) for (i, k) of Subproblem i by (16) and similarly (14e) for (i, k) of Subproblem k by (17). If we apply the same logic to all edges connecting to bus i , we can construct a smaller feasible region $\hat{\mathcal{C}}_i$ for Subproblem i . For the edge (i, k) , the constructions of $\hat{\mathcal{C}}_i$ and $\hat{\mathcal{C}}_k$ can help $W_{ik}^{(i)}$ and $W_{ik}^{(k)}$ converge to W_{ik}^* faster.

With this modification, the progress of the algorithm for the five-bus example is also depicted in Fig. 5(b), where we can see that the algorithm converges faster.

2) *Feasible Solution Generation*: When the algorithm converges, we have that

$$\text{Tr}(\mathbf{A}^{(i)} \mathbf{W}^{(i)}) = \text{Tr}(\tilde{\mathbf{A}}^{(i)} \mathbf{W}^{(i)}), \quad \forall i, \quad (18)$$

which holds when all its associated λ_{ik} 's are optimal; this is equivalent to have both of the following held:

$$\text{Tr}(\mathbf{A}^{(i)} \mathbf{W}^{(i)}) = \text{Tr}(\mathbf{A}^{(i)} \mathbf{W}^{(i)*}) \Leftrightarrow P_i = P_i^*, \quad \forall i, \quad (19)$$

$$\text{Tr}(\mathbf{B}^{(i)} \mathbf{W}^{(i)}) = \text{Tr}(\mathbf{B}^{(i)} \mathbf{W}^{(i)*}) \Leftrightarrow Q_i = Q_i^*, \quad \forall i. \quad (20)$$

In other words, Algorithm 1 tries to find the the optimal active and reactive power pair $[P_i^*, Q_i^*]^T$ for each bus i by manipulating λ_{ik} 's defined for the corresponding lines. The more lines are connected to a bus (i.e., the more λ_{ik} 's it involves), the more difficult is for (19) and (20) to hold. The $[P_i, Q_i]^T$ pair affects the $[P_k, Q_k]^T$ pair through λ_{ik} . Consider the situation where edge (i, k) is the only line connected to bus k except for bus i . When $[P_k, Q_k]^T$ becomes optimal, this helps bus i converge in the sense that this reduces the variations of $[P_i, Q_i]^T$ induced from bus k . When Algorithm 1 evolves, the $[P_k, Q_k]^T$ of leaf bus² k converges first as a leaf bus has only one edge. Then, we have the buses connected to the leaf buses converged. We continue this process and finally go up to the feeder.

For any leaf node k , we have $P_k = P_{ki}$ and $Q_k = Q_{ki}$, where bus i is the only bus connected to bus k . When the algorithm evolves, we obtain $W_{ik}^{(k)*}$ from the solution of the k th subproblem

²A leaf bus is a bus with one line attached only.

(14) when $\text{Tr}(\mathbf{A}^{(k)}\mathbf{W}^{(k)})$ and $\text{Tr}(\mathbf{B}^{(k)}\mathbf{W}^{(k)})$ are equal to P_k^* and Q_k^* , respectively. Once we have fixed $W_{ik}^{(k)*}$, we can add the constraint $W_{ik}^{(i)} = W_{ik}^{(k)*}$ to the i th subproblem for bus i by passing a message containing the value of $W_{ik}^{(k)*}$ from bus k to bus i . In matrix form, this constraint is equivalent to $\text{Tr}(\mathbf{C}^{(i)}\mathbf{W}^{(i)}) = \text{Re}\{W_{ik}^{(k)*}\}$ and $\text{Tr}(\mathbf{D}^{(i)}\mathbf{W}^{(i)}) = \text{Im}\{W_{ik}^{(k)*}\}$, where $\mathbf{C}^{(i)} = (C_{lm}^{(i)}, l, m \in \mathcal{N}_i)$, with $C_{lm}^{(i)} = \frac{1}{2}$ if $l = i$ and $m = k$, $C_{lm}^{(i)} = \frac{1}{2}$ if $l = k$ and $m = i$, and $C_{lm}^{(i)} = 0$ otherwise; and $\mathbf{D}^{(i)} = (D_{lm}^{(i)}, l, m \in \mathcal{N}_i)$, with $D_{lm}^{(i)} = \frac{1}{2}j$ if $l = i$ and $m = k$, $D_{lm}^{(i)} = -\frac{1}{2}j$ if $l = k$ and $m = i$, and $D_{lm}^{(i)} = 0$ otherwise. In this case, we reduce the n -bus network into the $(n-1)$ -bus one by removing bus k . When all other buses with positive active power flown from bus i (i.e. $\{l : l \sim i, l > i\}$) have been fixed and “removed”, bus i becomes a leaf bus in the reduced network. This process continues until we find all $W_{lm}^*, \forall (l, m) \in \mathcal{E}$. The global solution \mathbf{W}^* can be constructed from those W_{lm}^* ’s. However, for any bus k , if we fix P_k and Q_k which is not optimal, these errors will make its connecting bus i being fixed afterwards result in incorrect P_i and Q_i , which are not optimal either. To achieve this, we observe $P_i[t]$ and $Q_i[t]$ for a certain time period and check if their variations are significant. Assume that we are at time t , for the active power, we can keep track of the previous T P_i ’s and the current $P_i[t]$, i.e. $[P_i[t-T], P_i[t-T+1], \dots, P_i[t]]^T$. We can say that $P_i[t]$ has been converged if its cumulative change is less than a certain threshold γ (e.g. 10^{-4}), i.e.,

$$\sum_{k=0}^{T-1} \frac{|P_i[t-T+k] - P_i[t-T+k+1]|}{|P_i[t-T+k]|} < \gamma; \quad (21)$$

with a similar condition for the reactive power.

3) *Hot Start*: The problem needs to be solved repeatedly; when there are changes to the active/reactive limits at any bus, we apply Algorithm 1 to the problem again. In each update, we usually have small variation between the new \bar{P}_i and the previous ones and also for \underline{P}_i . Thus, in subsequent instances of the problem, the optimal angle difference across each line usually does not vary significantly. Therefore, we can set $\lambda_{ik}[0]$ with the optimal λ_{ik}^* which can be determined from the previous optimal W_{ik}^* .

VI. CASE STUDIES

We test the algorithm performance on the 34- and 123-bus test feeder systems [23]. Assume that the system operates under nominal conditions specified by the given dataset, for one day

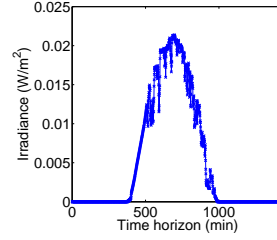


Fig. 6. Irradiance of a particular day

(1440 minutes) with constant load demands $\hat{P}_i, \forall i$. We assume that on each bus i , there are an energy storage and connected photovoltaics (PV)-based electricity generation resources, which can supply active power, P_i^{PV} to the bus locally, i.e. their net effect is to reduce the load. If all P_i^{PV} is consumed locally, then the active power injection at bus i will be $\bar{P}_i = \hat{P}_i + P_i^{PV} \leq 0$. The computed optimal $P_i^* \in [\hat{P}_i, \bar{P}_i], i = 2, \dots, n$, will then be adjusted by controlling the amount of power from the PV devices which will be stored at the local storage. Let \hat{Q}_i be the nominal reactive power injection at bus i . It turns out the power electronics on modern PV panels can supply reactive power in a range that is sufficient to cancel the nominal reactive power [24], so we assume that the reactive power can be adjusted in the ranges specified by i) $Q_i \in [0, 1.2\hat{Q}_i]$, if $\hat{Q}_i \geq 0$, and ii) $Q_i = [-1.2\hat{Q}_i, 0]$ otherwise.

We consider irradiance data in Fig. 6, which correspond to a particular day in November 2011 collected at the University of Nevada [25]; the data are sampled every minute. Assume that the PV module at bus i can provide up to 20% of its nominal load \hat{P}_i . Thus P_i^{PV} 's at different buses are different and are proportional to their respective \hat{P}_i . At different time, P_i^{PV} varies according to the irradiance data.

Since there is only radiation between the 377th and 991th minutes, we define the time horizon as $[377, 991]$. We execute the algorithm every one-minute cycle, corresponding to the PV generation changes in this period. Recall that the algorithm requires inputs of Lagrangian multipliers as the starting points. In minute t , where $t \in [377, 991]$, the inputs to the algorithm are the Lagrangian multipliers computed by the algorithm at time $t - 1$. Moreover each Lagrangian multiplier is only stored and manipulated by the two buses at the two ends of the corresponding line. The initial Lagrangian multipliers, i.e. at $t = 377$, are computed from the nominal system settings.

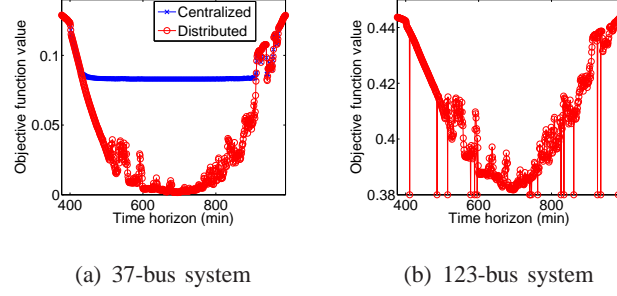


Fig. 7. Objective function values of the solutions computed by the distributed algorithm

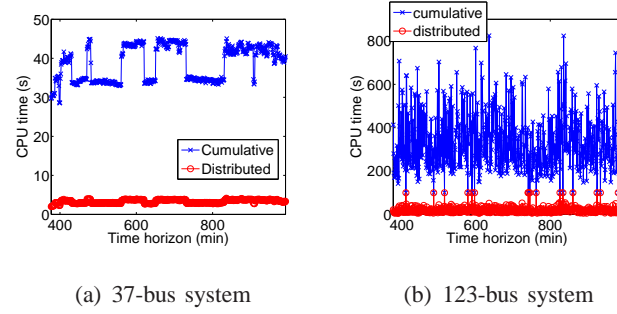


Fig. 8. Computation time of the distributed algorithm

In order to check if the distributed algorithm can obtain the global optimums, we compare the objective function values computed by the distributed algorithm and those by the centralized solver; the results are plotted in Fig. 7. For the 37-bus system, we can see that the distributed algorithm converges to the optimums all the time, whereas for the 123-bus system convergence occurs most of the time. In the 37-bus and 123-bus systems, the centralized solver failed to solve the system due to convergence issues. Fig. 8 shows the corresponding computational times elapsed. Here we only consider the CPU time spent on the SDP solver and assume that communication overheads can be neglected. In our simulation, we implement the algorithm iteratively; in each iteration, we solve the subproblems sequentially. In Fig. 8, each subfigure contains two curves. One (distributed) is to sum the CPU times of the subproblems which need the longest CPU time in each iteration. In other words, we only consider the most demanding subproblem in each iteration and then sum the CPU times spent on these subproblems in all iterations. The average CPU times for the three cases are 1.28 s, 3.33 s, and 19.69 s, respectively,

and they are short enough when each cycle lasts one minute.

VII. CONCLUDING REMARKS

We proposed to solve a convex optimization based method to solve the voltage regulation problem in distribution networks. We cast the problem as a loss minimization program. We showed that under broad conditions that are likely to be satisfied in practice, the optimization problem can be solved via its convex relaxation. We then proposed a asynchronous and distributed algorithm that can be implemented in a network with a large number of buses; we demonstrated the effectiveness of the algorithm with two case studies.

APPENDIX

Proof of Theorem 1: The first and third cases are clear. The interesting case to prove is to show that if \mathbf{W}^* has rank higher than 1, there are no rank 1 \mathbf{W} that is feasible.

The requirement that $\underline{Q}_i < \beta_i$ for $i = 2, \dots, n$ is to ensure that the reactive lower bound is in fact never tight for all the nodes in the network. Let h be parent of i and k be a child of i . Since we assume that power always flow from parent to child in the network, and from the angle constraint in (9), $0 \leq \theta_{hi} \leq \tan^{-1}(\frac{b_{ik}}{g_{ik}})$. Over this range, $Q_{ih} \geq 0$. This corresponds to the intuition that reactive power should flow up the tree to support the voltage. Note the inductive line is very lossy in terms of reactive powers, therefore i might receive or supply reactive power to k . The Q_{ik} is monotonic in θ_{ik} starting at $\theta_{ik} = 0$ until it reaches its minimum at an angle of $\tan^{-1}(\frac{g_{ik}}{b_{ik}})$. Let $\tilde{\theta}_{ik} = \min(\tan^{-1}(\frac{g_{ik}}{b_{ik}}), \bar{\theta}_{ik})$, then $Q_i = Q_{ih} + \sum_{k:k \in \mathcal{C}(i)} Q_{ik} \geq \sum_{k:k \in \mathcal{C}(i)} Q_{ik} \geq \sum_{k:k \in \mathcal{C}(i)} b_{ik} - g_{ik} \sin(\tilde{\theta}_{ik}) - b_{ik} \cos(\theta_{ik}) = \beta_i$. Therefore if $\underline{Q}_i < \beta_i$, the reactive power lower bounds are never tight.

To finish the proof we need to introduce some notations from [14]. For a n bus network, let $\{\bar{\theta}_{ik}\}$ be the set of angle constraints, one for each line. Then, the angle-constrained *active power injection region* is the set of all active power injection vectors that satisfy the line angle constraints, i.e., $\mathcal{P}_\theta = \{\mathbf{p} : \mathbf{p} = \text{Re}\{\text{diag}(\mathbf{v}\mathbf{v}^H \mathbf{Y}^H)\}, |V_i| = 1, |\theta_{ik}| \leq \bar{\theta}_{ik}\}$. Let $\mathcal{F}_\theta \subset \mathbb{R}^{2n-2}$ be the Cartesian product region of the $n-1$ active line flow regions, then $\mathcal{F}_\theta = \prod_{i \sim k} \mathcal{F}_{\theta,ik}$. Let $\mathbf{M} \in \mathbb{R}^{n \times 2n-2}$ be a matrix with the rows indexed by the buses and the columns indexed by the $2(n-1)$ ordered pair of edges, i.e., if i is connected to k , both (i, k) and (k, i) are included; thus $\mathbf{M}[i, (k, l)] = 1$ if $i = k$ or $i = l$, and $\mathbf{M}[i, (k, l)] = 0$ otherwise. \mathbf{M} is a generalized edge

to bus incidence matrix, and $\mathcal{P}_\theta = \mathbf{M}\mathcal{F}_\theta$ i.e., the power injection region is obtained by a linear transformation of the product of line flow regions.

Similarly, \mathcal{G}_θ is the product region of the $n - 1$ reactive line flow regions. Then, for all (i, k) , by stacking the \mathbf{H}_{ik} 's as defined in (11) into a $2(n - 1) \times 2(n - 1)$ block diagonal matrix \mathbf{H} , i.e., $\mathbf{H} = \text{diag}(\{H_{ik}\}_{i \sim k})$, we obtain a the global transform between \mathcal{F} and \mathcal{G} . The angle-constrained reactive power injection region \mathcal{Q}_θ is given by $\mathcal{Q}_\theta = \mathbf{M}\mathcal{G}_\theta = \mathbf{M}\mathbf{H}\mathcal{F}_\theta$.

Since the lower reactive power injections bounds are never tight by construction, we ignore them from now on. Let \mathcal{P} be the feasible region of the original problem (6), that is, $\mathcal{P} = \{\mathbf{p} : \exists \mathbf{v} \in \mathbb{C}^n, P_i = \text{Tr}(\mathbf{A}_i \mathbf{v} \mathbf{v}^H), |V_i| = 1, \underline{P}_i \leq P_i \leq \overline{P}_i, \text{Tr}(\mathbf{B}_i \mathbf{v} \mathbf{v}^H) \leq \overline{Q}_i, |\theta_{ik}| < \overline{\theta}_{ik}, \forall i \sim k\}$. We can equivalently write \mathcal{P} as $\mathcal{P} = \mathbf{M}(\mathcal{F}_\theta \cap \mathcal{F}_P \cap \mathcal{F}_Q)$, where \mathcal{F}_P is the flow region satisfying the real power constraints, that is, $\mathcal{F}_P = \{\mathbf{f} \in \mathbb{R}^{2n-2} : \mathbf{p} = \mathbf{M}\mathbf{f}, \underline{P}_i \leq P_i \leq \overline{P}_i\}$. \mathcal{F}_Q is the flow region satisfying the reactive power constraints, that is, $\mathcal{F}_Q = \{\mathbf{f} \in \mathbb{R}^{2n-2} : \mathbf{q} = \mathbf{M}\mathbf{H}\mathbf{f}, Q_i \leq \overline{Q}_i\}$. Since \mathcal{F}_P and \mathcal{F}_Q are defined by linear inequalities, they are convex. However, \mathcal{F}_θ is not.

Let \mathcal{S} be the feasible region of the relaxed problem (8). It turns out that $\mathcal{S} = \mathbf{M}(\text{convhull}(\mathcal{F}_\theta) \cap \mathcal{F}_P \cap \mathcal{F}_Q)$, is convex, and contains \mathcal{P} .

Now we need to define the Pareto-front of a set. Let $\mathcal{X} \subset \mathbb{R}^n$, we say $\mathbf{x} \in \mathcal{X}$ is Pareto-optimal if $\nexists \mathbf{y} \in \mathcal{X}$ such that $\mathbf{y} \leq \mathbf{x}$ with strict inequality in at least one coordinate. The set of Pareto-optimal points is called the Pareto-front of \mathcal{X} , and labeled $\mathcal{O}(\mathcal{X})$. When minimizing a strictly increasing function, the optimal is always achieved in the Pareto-front. Therefore to show the second statement in the theorem, it suffices to show the following lemma,

Lemma 2. *Suppose \mathcal{P} is not empty, then $\mathcal{P} = \mathcal{O}(\mathcal{S})$.*

Suppose the lemma is true, then if the optimal solution of the relaxed problem (8) is of rank 2, then \mathcal{P} must be empty.

The proof of this lemma is similar to Lemma 4 in [14]. Let $\mathbf{p}^* \in \mathcal{S}$ be the optimal solution of the relaxed problem, $\mathbf{f}^* \in \text{convhull}(\mathcal{F}_\theta) \cap \mathcal{F}_P \cap \mathcal{F}_Q$ its corresponding active flow vector and $\mathbf{r}^* = \mathbf{H}\mathbf{f}^*$ be the corresponding reactive power flow vector. It suffices to show that if $P_i^* > \underline{P}_i$, then $(f_{ik}^*, f_{ki}^*) \in \mathcal{F}_{\theta,ik}$ for every $k \sim i$. Once this fact is established, the rest of the proof is the same as the proof of Lemma 4 in [14]. Suppose that $P_i^* > \underline{P}_i$, but $(f_{ik}^*, f_{ki}^*) \notin \mathcal{F}_{\theta,ik}$ for some k . Then there exists $\epsilon > 0$ such that $(f_{ik}^* - \epsilon, f_{ki}^*) \in \text{convhull}(\mathcal{F}_{\theta,ik})$. Let $(\tilde{f}_{ik}, \tilde{f}_{ki}) = (f_{ik}^* - \epsilon, f_{ki}^*)$.

Since

$$\mathbf{H}_{ik} \begin{bmatrix} -\epsilon \\ 0 \end{bmatrix} = -\frac{\epsilon}{2b_{ik}g_{ik}} \begin{bmatrix} b_{ik}^2 - g_{ik}^2 \\ b_{ik}^2 + g_{ik}^2 \end{bmatrix} < \mathbf{r}^*,$$

Therefore $(\tilde{f}_{ik}, \tilde{f}_{ki})$ is a better feasible flow on the line (i, k) , which contradicts the optimality of \mathbf{f}^* . ■

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